$P-$FERRER DIAGRAM, $P-$LINEAR IDEALS AND ARITHMETICAL RANK

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Abstract

In this paper we introduce $p-$Ferrer diagram, note that $1-$ Ferrer diagram are the usual Ferrer diagrams or Ferrer board, and corresponds to planar partitions. To any $p-$Ferrer diagram we associate a $p-$Ferrer ideal. We prove that $p-$Ferrer ideal have Castelnuovo mumford regularity $p+1$. We also study Betti numbers , minimal resolutions of $p-$Ferrer ideals. Every $p-$Ferrer ideal is $p-$joined ideals in a sense defined in a fortcoming paper [M], which extends the notion of linearly joined ideals introduced and developped in the papers [BM2], [BM4],[EGHP] and [M]. We can observe the connection between the results on this paper about the Poincaré series of a $p-$Ferrer diagram $\Phi$and the rook problem, which consist to put k rooks in a non attacking position on the $p-$Ferrer diagram Φ.

1 Introduction

We recall that any non trivial ideal $\mathcal{I} \subset S$ has a finite free resolution :

$$0 \to F_s \overset{M_s}{\to} F_{s-1} \to .... \to F_1 \overset{M_1}{\to} \mathcal{I} \to 0$$

the number s is called the projective dimension of $S/\mathcal{I}$ and the Betti numbers are defined by $\beta_i(\mathcal{I}) = \beta_{i+1}(S/\mathcal{I}) = rankF_{i+1}$. By the theorem of Auslander and Buchsbaum we know that $s = \dim S -$ depth $(S/\mathcal{I})$. We will say that the ideal $\mathcal{I}$ has a pure resolution if $F_i = S^{\beta_i}(-a_i)$ for all $i = 1, ..., s$. This means that $\mathcal{I}$ is generated by elements in degree a_1, and for $i \geq 2$ the matrices M_i in the minimal free resolution of $\mathcal{I}$ have homogeneous entries of degree $a_i - a_{i-1}$.

We will say that the ideal $\mathcal{I}$ has a $p-$linear resolution if its minimal free resolution is linear, i.e. $\mathcal{I}$ has a pure resolution and for $i \geq 2$ the matrices M_i have linear entries.

If $\mathcal{I}$ has a pure resolution, then the Hilbert series of $S/\mathcal{I}$ is given by:

$$H_{S/\mathcal{I}}(t) = \frac{1 - \beta_1 t^{a_1} + ... + (-1)^s \beta_s t^{a_s}}{(1-t)^n}$$

where $n = \dim S$. Since $a_1 < ... < a_s$ it the follows that if $\mathcal{I}$ has a pure resolution then the Betti numbers are determined by the Hilbert series.

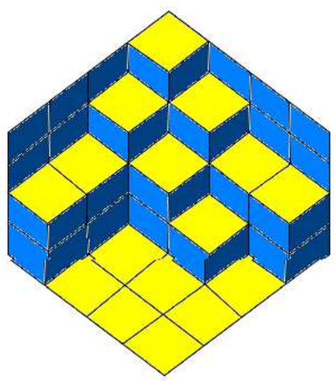
$p-$**Ferrer partitions and diagrams.** The $1-$Ferrer partition is a nonzero natural integer λ, a $2-$Ferrer partition is called a partition and is given by a sequence $\lambda_1 \geq ... \geq \lambda_m > 0$

of natural integers, a $3-$Ferrer partition is called planar partition. $p-$Ferrer partitions are defined inductively $\Phi : \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$, where λ_j is a $p-1$Ferrer partition for $j = 1, ..., m$, and the relation $\leq$ is also defined recursively: if $\lambda_i : \lambda_{i,1} \geq ... \geq \lambda_{i,s}, \lambda_{i+1} : \lambda_{i+1,1} \geq ... \geq \lambda_{i+1,s'}$ we will say that $\lambda_i \geq \lambda_{i+1}$ if and only if $s \geq s'$ and $\lambda_{i,j} \geq \lambda_{i+1,j}$ for any $j = 1, ..., s'$. Up to my knowledge there are very few results for $p-$Ferrer partitions in bigger dimensions.

To any $p-$Ferrer partition we associate a $p-$Ferrer diagram which are subsets of $I\!N^p$. The 1-Ferrer diagram associated to $\lambda \in I\!N$ is the subset $\{1, ..., \lambda\}$. Inductively if $\Phi : \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$, is a $p-$Ferrer partition, where λ_j is a $p-1$-Ferrer partition for $j = 1, ..., m$, we associate to Φ the $p-$Ferrer diagram $\Phi = \{(\eta, 1), \eta \in \lambda_1\} \cup ... \cup \{(\eta, m), \eta \in \lambda_m\}$. Ferrer $p-$diagrams can also be represented by a set of boxes labelled by a $p-$uple $(i_1, ..., i_p)$ of non zero natural numbers, they have the property that if $1 \leq i'_1 \leq i_1, ..., 1 \leq i'_p \leq i_p$, then the box labelled $(i'_1, ..., i'_p)$ is also in the $p-$Ferrer diagram. We can see that for two Ferrer diagrams: $\Phi_1 \geq \Phi_2$ if and only if the set of boxes of Φ_1 contains the set of boxes of Φ_2.

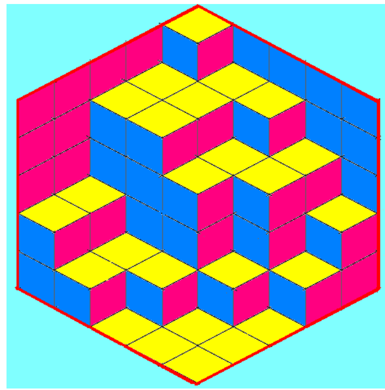
Example 1 *The following picture corresponds to the $3-$Ferrer diagram given by:*

$$\begin{matrix} 4 & 3 & 2 & 2 \\ 3 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{matrix}$$



Example 2 *The following picture corresponds to the $3-$Ferrer diagram given by:*

$$\begin{matrix} 5 & 4 & 4 & 3 & 2 \\ 4 & 4 & 3 & 3 & 1 \\ 4 & 4 & 3 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{matrix}$$



Definition 1 *Given a $p-$Ferrer diagram (or partition) Φ we can associated a monomial ideal $\mathcal{I}_\Phi$ in the following way. Let consider the polynomial ring $K[\underline{x}^{(1)}, \underline{x}^{(2)}, ..., \underline{x}^{(p)}]$ where $\underline{x}^{(i)}$ stands for the infinitely set of variables :$\underline{x}^{(i)} = \{x_1^{(i)}, x_2^{(i)}, ...\}$, we define inductively the ideal $\mathcal{I}_\Phi$*

1. *For $q = 2$ let $\Phi : \lambda \in I\!N^*$, then $\mathcal{I}_\Phi$ is the ideal generated by the variables $x_1^{(1)}, x_2^{(1)}, ..., x_\lambda^{(1)}$.*
2. *For $q = 2$ let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$ be a $2-$Ferrer diagram, then $\mathcal{I}_\Phi$ is an ideal in the ring of polynomials $K[x_1, ..., x_m, y_1, ..., y_{\lambda_1}]$ generated by the monomials $x_i y_j$ such that $i = 1, ..., m$ and $j = 1, ..., \lambda_i$. In this case $x_j^{(1)} = y_j, x_j^{(2)} = x_i$.*
3. *For $p > 2$ let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$ be a Ferrer diagram, where λ_j is a $p-1$Ferrer diagram. Let $\mathcal{I}_{\lambda_j} \subset K[\Lambda]$ be the ideal associated to λ_j, where $K[\Lambda]$ s a polynomial ring in a finite set of variables then $\mathcal{I}_\Phi$ is an ideal in the ring of polynomials $K[x_1^{(p)}, ..., x_m^{(p)}, \Lambda]$ generated by the monomials $x_i^{(p)} y_j$ such that $i = 1, ..., m$ and $y_j \in \mathcal{I}_{\lambda_i}$. That is*

$$\mathcal{I}_\Phi = (\bigcup_{i=1}^{m} \{x_i^{(p)}\} \times \mathcal{I}_{\lambda_i}).$$

We can observe the connection between the results on this paper about the Poincaré series of a $p-$Ferrer diagram $\Phi$and the rook problem, which consist to put k rooks in a non attacking position on the $p-$Ferrer diagram Φ. This will be developped in a forthcoming paper.

2 $p-$Ferrer' ideals

Lemma 1 *Let S be a polynomial ring, $\Gamma_2 ..., \Gamma_r$ be non empty disjoint sets of variables, set $\mathcal{A}_i$ the ideal generated by $\Gamma_{i+1}, ..., \Gamma_r$. Let $\mathcal{B}_2 \subset ... \subset \mathcal{B}_r$ be a sequence of ideals (not necessarily distinct), generated by the sets $B_2 \subset ... \subset B_r$. We assume that no variable of $\Gamma_2 \cup ... \cup \Gamma_r$ appears in $B_2, ..., B_r$, then*

$$\mathcal{A}_1 \cap (\mathcal{A}_2, \mathcal{B}_2) \cap ... \cap (\mathcal{B}_r) = (\bigcup_{2}^{r} \Gamma_i \times B_i)$$

where for two subsets $A, B \subset S$, we have set $A \times B = \{a\ b \mid a \in A,\ b \in B\}$.

Proof Let remark that if Γ is a set of variables and $P \subset S$ is a set of polynomials such that no variable of Γ appears in the elements of P then $(\Gamma) \cap (P) = (\Gamma \times P)$. Moreover if Γ_1, Γ_2 are disjoint sets of variables and $P \subset S$ is a set of polynomials such that no variable of Γ_1, Γ_2 appears in the elements of P then $(\Gamma_1, \Gamma_2) \cap (\Gamma_1, , P) = (\Gamma_1, \Gamma_2 \times P)$.

We prove by induction on the number k the following statement:

$$\mathcal{A}_1 \cap (\mathcal{A}_2, \mathcal{B}_2) \cap ... \cap (\mathcal{A}_k, \mathcal{B}_k) = (\mathcal{A}_k, \bigcup_2^k \Gamma_i \times B_i).$$

If $k = 2$, it is clear that $\Gamma_2 \times B_2 \subset \mathcal{A}_1 \cap (\mathcal{A}_2, \mathcal{B}_2)$, now let $f \in \mathcal{A}_1 \cap (\mathcal{A}_2, \mathcal{B}_2)$, we can write $f = f_1 + f_2$, where $f_1 \in (A_2)$, $f_2 \in (\Gamma_2)$ and no variable of A_2 appears in f_2, it follows that $f_2 \in (\Gamma_2) \cap (B_2) = (\Gamma_2 \times B_2)$.

Suppose that

$$\mathcal{A}_1 \cap (\mathcal{A}_2, \mathcal{B}_2) \cap ... \cap (\mathcal{A}_k, \mathcal{B}_k) = (\mathcal{A}_k, \bigcup_2^k \Gamma_i \times B_i),$$

we will prove that

$$\mathcal{A}_1 \cap (\mathcal{A}_2, \mathcal{B}_2) \cap ... \cap (\mathcal{A}_{k+1}, \mathcal{B}_{k+1}) = (\mathcal{A}_{k+1}, \bigcup_2^{k+1} \Gamma_i \times B_i).$$

Since $\Gamma_i \subset \mathcal{A}_j$, for $j < i$, and $B_i \subset B_j$ for $i \leq j$, we have $\bigcup_2^{k+1} \Gamma_i \times B_i \subset (\mathcal{A}_j, \mathcal{B}_j)$ for $1 \leq j \leq k$, so we have the inclusion " $\supset$ ".

By induction hypothesis we have that

$$\mathcal{A}_1 \cap (\mathcal{A}_2, \mathcal{B}_2) \cap ... \cap (\mathcal{A}_{k+1}, \mathcal{B}_{k+1}) = (\mathcal{A}_k, \bigcup_2^k \Gamma_i \times B_i) \cap (\mathcal{A}_{k+1}, \mathcal{B}_{k+1}).$$

Now let $f \in (\mathcal{A}_k, \bigcup_2^k \Gamma_i \times B_i) \cap (\mathcal{A}_{k+1}, \mathcal{B}_{k+1})$. we can write $f = f_1 + f_2 + f_3$, where $f_3 \in (\bigcup_2^k \Gamma_i \times B_i) \subset \mathcal{B}_{k+1}$, $f_1 \in \mathcal{A}_{k+1}$, and $f_2 \in (\Gamma_{k+1})$, and no variable of $\Gamma_{k+1} \cup ... \cup \Gamma_r$ appears in f_2, this would imply that $f_2 \in (\Gamma_{k+1}) \cap \mathcal{B}_{k+1} = (\Gamma_{k+1} \times B_{k+1})$.

Definition 2 *Let $\lambda_{m+1} = 0, \delta_0 = 0$, δ_1 be the highest integer such that $\lambda_1 = ... = \lambda_{\delta_1}$, and by induction we define δ_{i+1} as the highest integer such that $\lambda_{\delta_i+1} = ... = \lambda_{\delta_{i+1}}$, and set l such that $\delta_{l-1} = m$. For $i = 0, ..., l-2$ let*

$$\Delta_{l-i} = \{x^{(p)}_{\delta_i+1}, ..., x^{(p)}_{\delta_{i+1}}\}, \mathcal{P}_{l-i} = \mathcal{I}_{\lambda_{\delta_{i+1}}}.$$

So we have: $\Phi = \{(\eta,1), \eta \in \lambda_1\} \cup ... \cup \{(\eta,m), \eta \in \lambda_m\}$ and

$$\mathcal{I}_\Phi = (\bigcup_{i=2}^{l} \Delta_i \times P_i) = (\bigcup_{i=1}^{m} \{x_i^{(p)}\} \times \mathcal{I}_{\lambda_i}).$$

where for all i, P_i is a set of generators of $\mathcal{P}_i$.

The following Proposition is an immediate consequence of the above lemma :

Proposition 1 *1. We have the following decomposition (probably redundant):*

$$\mathcal{I}_\Phi = (x_1^{(p)}, ..., x_m^{(p)}) \cap (x_1^{(p)}, ..., x_{m-1}^{(p)}, \mathcal{I}_{\lambda_m}) ... \cap (x_1^{(p)}, ..., x_{i-1}^{(p)}, \mathcal{I}_{\lambda_i}) \cap ... (\mathcal{I}_{\lambda_m}),$$

2. Let $\mathcal{D}_i = (\bigcup_{j=i+1}^{l} \Delta_j)$, and $\mathcal{Q}_i = (\mathcal{D}_i, \mathcal{P}_i)$. Then

$$\mathcal{I}_\Phi = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap ... \cap \mathcal{Q}_l.$$

3. The minimal primary decomposition of $\mathcal{I}_\Phi$ is obtained inductively. Let $\mathcal{I}_{\lambda_{\delta_i}} = \mathcal{Q}_1^{(i)} \cap ... \cap \mathcal{Q}_{r_i}^{(i)}$ be a minimal prime decomposition, where by induction hypothesis $\mathcal{Q}_j^{(i)}$ is a linear ideal, then the minimal prime decomposition of $\mathcal{I}_\Phi$ is obtained from this decomposition by putting out unnecessary components.

Example 3 *let $\mathcal{P}_2 = (c,d) \cap (e)$, $\mathcal{P}_3 = (c,d) \cap (c,e) \cap (e,f)$ and*

$$\mathcal{I}_\Phi = (a,b) \cap (a, \mathcal{P}_2) \cap \mathcal{P}_3$$

then

$$\mathcal{I}_\Phi = (a,b) \cap (a,e) \cap (c,d) \cap (c,e) \cap (e,f)$$

is its minimal prime decomposition.

Proposition 2 *Let $\mathcal{I} \subset R$ be a $p-$Ferrer ideal then $\operatorname{reg}(\mathcal{I}) = p = \operatorname{reg}(R/\mathcal{I}) + 1$.*

Proof For any two ideals $\mathcal{J}_1, \mathcal{J}_2 \subset S$ we have the following exact sequence:

$$0 \to S/\mathcal{J}_1 \cap \mathcal{J}_2 \to S/\mathcal{J}_1 \oplus S/\mathcal{J}_2 \to S/(\mathcal{J}_1 + \mathcal{J}_2) \to 0$$

From [B-S, p. 289]

$$\operatorname{reg}(S/\mathcal{J}_1 \cap \mathcal{J}_2) \leq \max\{\operatorname{reg}(S/\mathcal{J}_1 \oplus S/\mathcal{J}_2), \operatorname{reg}(S/(\mathcal{J}_1 + \mathcal{J}_2)) + 1\}$$

in our case we take $\mathcal{J}_1 = \bigcap_{i=1}^{k} \mathcal{Q}_i, \mathcal{J}_2 = \mathcal{Q}_{k+1}$, so that $\operatorname{reg}(S/(\bigcap_{i=1}^{k} \mathcal{Q}_i + \mathcal{Q}_{k+1})) = \operatorname{reg}(S/(\mathcal{D}_k + \mathcal{P}_{k+1})) = \operatorname{reg}(S'/(\mathcal{P}_{k+1})) = p-1$, where $S = S'[\mathcal{D}_k]$. It then follows that $\operatorname{reg}(S/(\bigcap_{i=1}^{l} \mathcal{Q}_i)) \leq p$, on the other hand $(\bigcap_{i=1}^{l} \mathcal{Q}_i))$ is generated by elements of degree p, this implies $\operatorname{reg}(S/(\bigcap_{i=1}^{l} \mathcal{Q}_i)) = p$.

We will show that in fact projdim $(S/\mathcal{I}_\lambda)$ is the number of diagonals in a $p-$Ferrer diagram.

Definition 3 *Let $\Phi : \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$ be a $p-$Ferrer diagram. We will say that the monomial in the $p-$Ferrer ideal (or diagram) $x^{(p)}_{\alpha_p} x^{(p-1)}_{\alpha_{p-1}} ... x^{(1)}_{\alpha_1}$ is in the $\alpha_p + \alpha_{p-1} + ... + \alpha_1 - p + 1$ diagonal. Let $s_\Phi(k)$ be the number of elements in the $k-$diagonal of Φ, we will say that the $k-$diagonal of Φ is full if $s_\Phi(k) = \binom{k-1+p-1}{p-1}$, which is the number of elements in the $k-$diagonal of $\mathbb{N}^p$, let remark that by the definition of $p-$Ferrer diagram if the $k-$diagonal of Φ is full then the $j-$diagonal of Φ is full for all $j = 1, ..., k$.*

Lemma 2 *1. We have the formula*

$$s_\Phi(k) = \sum_{i=1}^{m} s_{\lambda_i}(k - (i-1)),$$

2. Let $df(\Phi)$ be the number of full diagonals of Φ, then

$$df(\Phi) = \min\{df(\lambda_i) + i - 1 \ \mid \ i = 1, ..., m\}$$

3. Let $\delta(\Phi)$ be the number of diagonals of Φ, then

$$\delta(\Phi) = \max\{\delta(\lambda_i) + i - 1 \ \mid \ i = 1, ..., m\},$$

and $\delta(\Phi) = \max_{i=2}^{l}\{\delta(\mathcal{P}_i) + \dim \mathcal{D}_{i-1} - 1\}$.

Proof The first item counts the number of elements in the $k-$diagonal of Φ by counting all the $i-$ slice pieces. The second item means that the $k-$diagonal of Φ is full if and only if the $k-(i-1)-$diagonal of the $i-$ slice piece is full, and finally the third item means there is an element in the $k-$diagonal of Φ if and only if there is at least one element in the $k-(i-1)-$diagonal of the $i-$slide piece of Φ, for some i.

Remark that $\delta(\Phi) = \max_{i=2}^{l}\{\delta(\mathcal{P}_i) + \dim \mathcal{D}_{i-1} - 1\}$, since $\max_{i=1}^{\delta_1}\{\delta(\lambda_i) + i - 1\} = \delta(\lambda_1) + \delta_1 - 1 = \delta(\mathcal{P}_l) + \dim \mathcal{D}_{l-1} - 1$, $\max_{i=\delta_1+1}^{\delta_2}\{\delta(\lambda_i) + i - 1\} = \delta(\lambda_{\delta_1+1}) + \delta_1 + \delta_2 - 1 = \delta(\mathcal{P}_{l-1}) + \dim \mathcal{D}_{l-2} - 1$, and so on.

Theorem 1 *Let consider a $p-$Ferrer diagram Φ and its associated ideal $\mathcal{I}_\Phi$ in a polynomial ring S. Let $n = \dim S, c = \mathrm{ht}\mathcal{I}_\Phi$, $d = n - c$. For $i = 1, ..., d - \operatorname{depth} S/\mathcal{I}$, let s_{d-i} be the numbers of elements in the $c+i$ diagonal of Φ. Then :*

1. c the height of $\mathcal{I}_\Phi$ is equal to the number of full diagonals.

2. For $j \geq 1$ we have

$$\beta_j(S/\mathcal{I}_\Phi) = \binom{c+p-1}{j+p-1}\binom{j+p-2}{p-1} + \sum_{i=0}^{d-1} s_i \binom{n-i-1}{j-1}$$

3. $\operatorname{projdim}(S/\mathcal{I}_\Phi) = \delta(\Phi)$.

Proof

1. We prove the statement by induction on p, if $p = 1$ and $\Phi = \lambda \in I\!N$, then $\mathcal{I}_\phi = (x_1, ..., x_\lambda)$ is an ideal of height λ and $df(\lambda) = \lambda$. Now let $p \geq 2$, since

$$\mathcal{I}_\Phi = (x_1^{(p)}, ..., x_m^{(p)}) \cap (x_1^{(p)}, ..., x_{m-1}^{(p)}, \mathcal{I}_{\lambda_m}) ... \cap (x_1^{(p)}, ..., x_{i-1}^{(p)}, \mathcal{I}_{\lambda_i}) \cap ... (\mathcal{I}_{\lambda_m}),$$

we have that

$$\mathrm{ht}\mathcal{I}_\Phi = \min\{ \ \mathrm{ht}\mathcal{I}_{\lambda_i} + i - 1\},$$

by induction hypothesis $\mathrm{ht}\mathcal{I}_{\lambda_i} = df(\lambda_i)$ so

$$\mathrm{ht}\mathcal{I}_\Phi = \min\{df(\lambda_i) + i - 1\} = df(\Phi).$$

2. The proof is by induction on the number of generators $\mu(\mathcal{I}_\Phi)$ of the ideal $\mathcal{I}_\Phi$. The statement is clear if $\mu(\mathcal{I}_\Phi) = 1$.

Suppose that $\mu(\mathcal{I}_\Phi) > 1$. Let π be a generator of $\mathcal{I}_\Phi$ being in the last diagonal of Φ, so we can write $\pi = x_i^{(p)} g$ for some i, where $g \in \mathcal{I}_{\lambda_i}$ is in the last diagonal of λ_i. By definition of a $p-$Ferrer tableau, the ideal generated by all the generators of $\mathcal{I}_\Phi$ except $x_i^{(p)} g$ is a p-Ferrer ideal and we denoted it by $\mathcal{I}_{\Phi'}$.

In the example 1 we can perform several steps :

$$\begin{array}{cccc} 4 & 3 & 2 & 2 \\ 3 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{cccc} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{cccc} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{cccc} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\Phi \qquad\qquad \Phi' \qquad\qquad \Phi'' \qquad\qquad \Phi'''$$

let denote $\alpha_p := i$, so that

$$x_i^{(p)} g = x_{\alpha_p}^{(p)} x_{\alpha_{p-1}}^{(p-1)} ... x_{\alpha_1}^{(1)}$$

For any k and $1 \leq \beta < \alpha_k$ we have that $x_{\alpha_p}^{(p)} x_{\alpha_{p-1}}^{(p-1)} ... x_\beta^{(k)} ... x_{\alpha_1}^{(1)} \in \mathcal{I}_{\Phi'}$, so we have that

$$(\{x_1^{(p)}, ..., x_{\alpha_p - 1}^{(p)}\}, ..., \{x_1^{(1)}, ..., x_{\alpha_1 - 1}^{(1)}\}) \subset \mathcal{I}_{\Phi'} : x_{\alpha_p}^{(p)} ... x_{\alpha_1}^{(1)}.$$

On the other hand let $\Pi \in \mathcal{I}_{\Phi'} : x_{\alpha_p}^{(p)} ... x_{\alpha_1}^{(1)}$ a monomial, we can suppose that no variable in $(\{x_1^{(p)}, ..., x_{\alpha_p - 1}^{(p)}\}, ..., \{x_1^{(1)}, ..., x_{\alpha_1 - 1}^{(1)}\})$ appears in Π, so $\Pi x_{\alpha_p}^{(p)} ... x_{\alpha_1}^{(1)} \in \mathcal{I}_{\Phi'}$ implies that there is a generator of $\mathcal{I}_{\Phi'}$ of the type $x_{\beta_p}^{(p)} ... x_{\beta_1}^{(1)}$ such that $\beta_i \geq \alpha_i$ for all $i = 1, ..., p$, this is in contradiction with the fact that $x_{\alpha_p}^{(p)} x_{\alpha_{p-1}}^{(p-1)} ... x_{\alpha_1}^{(1)}$ is in the last diagonal of Φ and doesn't belongs to $\mathcal{I}_{\Phi'}$. In conclusion we have that

$$\mathcal{I}_{\Phi'} : x_{\alpha_p}^{(p)} ... x_{\alpha_1}^{(1)} = (\{x_1^{(p)}, ..., x_{\alpha_p - 1}^{(p)}\}, ..., \{x_1^{(1)}, ..., x_{\alpha_1 - 1}^{(1)}\})$$

is a linear ideal generated by $\alpha_p + ... + \alpha_1 - (p)$ variables. Let remark that since $x^{(p)}_{\alpha_p} x^{(p-1)}_{\alpha_{p-1}} ... x^{(1)}_{\alpha_1}$ is in the last diagonal the number of diagonals $\delta(\Phi)$ in Φ is $\alpha_p + ... + \alpha_1 - p + 1$.

We have the following exact sequence :

$$0 \to S/(\mathcal{I}_{\Phi'} : (x_i^{(p)} g))(-p) \xrightarrow{\times x_i^{(p)} g} S/(\mathcal{I}_{\Phi'}) \to S/(\mathcal{I}_{\Phi}) \to 0,$$

by applying the mapping cone construction we have that

$$\beta_j(S/\mathcal{I}_\Phi) = \beta_j(S/\mathcal{I}_{\Phi'}) + \binom{\delta(\Phi)-1}{j-1}, \quad \forall j = 1, ..., \ \operatorname{projdim}(S/\mathcal{I}_\Phi).$$

By induction hypothesis the number of diagonals in Φ' coincides with $\operatorname{projdim}(S/\mathcal{I}_{\Phi'})$. The number of diagonals in Φ' is either equal to the number of diagonals in Φ minus one, or equal to the number of diagonals in Φ. In both cases we have that $s_i(\Phi) = s_i(\Phi')$ for $i = d-1, ..., n-(\delta(\Phi)-1)$, $s_{n-(\delta(\Phi))}(\Phi') = s_{n-(\delta(\Phi))}(\Phi) - 1$, and $s_i(\Phi) = s_i(\Phi') = 0$ for $i < n-(\delta(\Phi))$.

Let $c' = \operatorname{ht}\mathcal{I}_{\Phi'}$, It then follows that

$$\beta_j(S/\mathcal{I}_{\Phi'}) = \binom{c'+p-1}{j+p-1}\binom{j+p-2}{p-1} + \sum_{i=0}^{d-1} s_i(\Phi')\binom{n-i-1}{j-1}, \ \forall j = 1, ..., \ \operatorname{projdim}(S/\mathcal{I}_{\Phi'}).$$

By induction hypothesis $\operatorname{projdim}(S/\mathcal{I}_{\Phi'}) = \delta(\Phi')$. We have to consider two cases:

(a) $\delta(\Phi) = c$, this case can arrive only if the c diagonal of Φ is full, so

$$c' = c-1, \quad s_{n-c}(\Phi') = \binom{c-1+p-1}{p-1} - 1, \delta(\Phi) = \delta(\Phi') = c$$

$$\forall 1 \leq j \leq c, \quad \beta_j(S/\mathcal{I}_\Phi) = \beta_j(c-1,p) + (\binom{c-1+p-1}{p-1} - 1)\binom{c-1}{j-1} + \binom{c-1}{j-1}.$$

$$\forall 1 \leq j \leq c, \quad \beta_j(S/\mathcal{I}_\Phi) = \beta_j(c-1,p) + \binom{c-1+p-1}{p-1}\binom{c-1}{j-1} = \beta_j(c,p).$$

Let remark that by induction hypothesis $\beta_j(S/\mathcal{I}_{\Phi'}) = 0$ for $j > c$, this implies that $\operatorname{projdim}(S/\mathcal{I}_\Phi) = c = \delta(\Phi)$.

(b) $\delta(\Phi) > c$, in this case $c' = c$

$$\beta_j(S/\mathcal{I}_\Phi) = \binom{c+p-1}{j+p-1}\binom{j+p-2}{p-1} + \sum_{i=0}^{d-1} s_i(\Phi)\binom{n-i-1}{j-1}$$

and $\operatorname{projdim}(S/\mathcal{I}_\Phi) = \operatorname{projdim}(S/\mathcal{I}_{\Phi'})$ equals the number of diagonals in Φ.

In particular it follows that if the number of diagonals in Φ' is equal to the number of diagonals in Φ minus one, $\operatorname{projdim}(S/\mathcal{I}_\Phi) = \operatorname{projdim}(S/\mathcal{I}_{\Phi'}) + 1$ is the number of diagonals in Φ. If the number of diagonals in Φ' is equal to the number of diagonals in Φ, then $\operatorname{projdim}(S/\mathcal{I}_\Phi) = \operatorname{projdim}(S/\mathcal{I}_{\Phi'})$ equals the number of diagonals in Φ.

Proposition 3 $\operatorname{ara}(\mathcal{I}_\Phi) = \operatorname{cd}(\mathcal{I}_\Phi) = \operatorname{projdim}(S/\mathcal{I}_\Phi)$.

Proof Recall that a monomial in the $p-$Ferrer ideal (or tableau) $x^{(p)}_{\alpha_p} x^{(p-1)}_{\alpha_{p-1}} ... x^{(1)}_{\alpha_1}$ is in the $\alpha_p + \alpha_{p-1} + ... + \alpha_1 - p + 1$ diagonal. Let $\mathcal{K}_j$ the set of all monomials in the Ferrer tableau lying in the j diagonal and let $F_j = \sum_{M \in \mathcal{K}_j} M$, we will prove that for any $M \in \mathcal{K}_j$, we have$M^2 \in (F_1, ..., F_j)$. If $j = 2$, let $M = x^{(p)}_{\alpha_p} x^{(p-1)}_{\alpha_{p-1}} ... x^{(1)}_{\alpha_1}$, with $\alpha_p + \alpha_{p-1} + ... + \alpha_1 - p + 1 = 2$, then

$$MF_2 = M^2 + \sum (x^{(p)}_{\alpha_p} x^{(p-1)}_{\alpha_{p-1}} ... x^{(1)}_{\alpha_1}) M'$$

One monomial $M' \in \mathcal{K}_2$, $M' \neq M$ can be written

$$x^{(p)}_{\beta_p} x^{(p-1)}_{\beta_{p-1}} ... x_{\beta^{(1)}_1}$$

with $\beta_p + \beta_{p-1} + ... + \beta_1 - p + 1 = 2$, this implies that $\beta_i = 1$ for all i except one value i_0, for which $\beta_{i_0} = 2$ and also $\alpha_j = 1$ for all j except one value j_0, for which $\alpha_{j_0} = 2$. Since $M' \neq M$ we must have $x^{(P)}_1 x^{(p-1)}_1 ... x^{(1)}_1$ divides MM'.

Now let $j \geq 3$, let $M = x^{(p)}_{\alpha_p} x^{(p-1)}_{\alpha_{p-1}} ... x^{(1)}_{\alpha_1}$, with $\alpha_p + \alpha_{p-1} + ... + \alpha_1 - p + 1 = j$, then

$$MF_2 = M^2 + \sum (x^{(p)}_{\alpha_p} x^{(p-1)}_{\alpha_{p-1}} ... x^{(1)}_{\alpha_1}) M'$$

One monomial $M' \in \mathcal{K}_j$, $M' \neq M$ can be written

$$x^{(p)}_{\beta_p} x^{(p-1)}_{\beta_{p-1}} ... x_{\beta^{(1)}_1}$$

with $\beta_p + \beta_{p-1} + ... + \beta_1 - p + 1 = j$, let i_0 such that $\beta_{i_0} \neq \alpha_{i_0}$ if $\beta_{i_0} < \alpha_{i_0}$ then $\dfrac{M}{x^{(i_0)}_{\alpha_{i_0}}} x^{(i_0)}_{\beta_{i_0}} \in K_i$ for some $i < j$, and if $\beta_{i_0} > \alpha_{i_0}$ then $\dfrac{M'}{x^{(i_0)}_{\beta_{i_0}}} x^{(i_0)}_{\alpha_{i_0}} \in K_i$ for some $i < j$,in both cases $MM' \in (K_i)$ for some $i < j$. As a consequence $\operatorname{ara}(\mathcal{I}_\Phi) \leq \operatorname{projdim}(S/\mathcal{I}_\Phi)$, but $\mathcal{I}_\Phi$ is a monomial ideal, so by a Theorem of Lyubeznik $\operatorname{cd}(\mathcal{I}_\Phi) = \operatorname{projdim}(S/\mathcal{I}_\Phi)$, and $\operatorname{cd}(\mathcal{I}_\Phi) \leq \operatorname{ara}(\mathcal{I}_\Phi)$, so we have the equality $\operatorname{ara}(\mathcal{I}_\Phi) = \operatorname{projdim}(S/\mathcal{I}_\Phi)$. Let remark that the equality $\operatorname{cd}(\mathcal{I}_\Phi) = \operatorname{projdim}(S/\mathcal{I}_\Phi)$ can be recovered by direct computations in the case of $p-$Ferrer ideals.

The reader should consider the relation between our theorem and the following result from [EG]:

Proposition 4 *If $R := S/\mathcal{I}$ is a homogeneous ring with $p-$linear resolution over an infinite field, and $x_i \in R_1$ are elements such that x_{i+1} is a non zero divisor on $(R/(x_1, ..., x_i))/H^0_{\mathbf{m}}(R/(x_1, ..., x_i)$, where $\mathbf{m}$ is the unique homogeneous maximal ideal of S, then*

1. *$s_i(R) = length(H^0_{\mathbf{m}}(R/(x_1, ..., x_i)_{p-1})$, for $i = 0, ..., \dim R - 1$.*
2. *If R is of codimension c, and $n := \dim S$, the Betti numbers of R are given by:*

$$\text{for} \quad j = 1, ..., n - \text{ depth}\,(R) \quad \beta_j(R) = \beta_j(c,p) + \sum_{i=0}^{d-1} s_i \binom{n-i-1}{j-1},$$

where $\beta_j(c,p) = \binom{c+p-1}{j+p-1}\binom{j+p-2}{p-1}$ are the betti numbers of a Cohen-Macaulay ring having $p-$linear resolution, of codimension c.

We have the following corollary:

Corollary 1 *If $R := S/\mathcal{I}$ is a homogeneous ring with $p-$linear resolution over an infinite field,of codimension c, and $n := \dim S$, then*

$$\beta_j(c,p) \leq \beta_j(R) \leq \beta_j(n - \text{ depth}\,(R), p).$$

Proof As a consequence of the above proposition we have that $s_i \leq \binom{n-(i+1)+p-1}{p-1}$ so that

$$\beta_j(c,p) \leq \beta_j(R) \leq \beta_j(c,p) + \sum_{i=0}^{d-1} \binom{n-(i+1)+p-1}{p-1}\binom{n-i-1}{j-1}$$

By direct computations we have that $\beta_j(c,p) + \binom{n-d+p-1}{p-1}\binom{n-d}{j\ell-1} = \beta_j(c+1,p)$, which implies

$$\beta_j(c,p) \leq \beta_j(R) \leq \beta_j(c+1,p) + \sum_{i=0}^{d-2} \binom{n-(i+1)+p-1}{p-1}\binom{n-i-1}{j-1},$$

by repeating the above computations we got the corollary.

3 Hilbert series of ideals with $p-$linear resolution.

Let $\mathcal{I} \subset S$ be an ideal with $p-$linear resolution, it follows from [EG], that the Hilbert series of $S/\mathcal{I}$ is given by

$$H_{S/\mathcal{I}}(t) = \frac{\sum_{i=0}^{p-1}\binom{c+i-1}{i}t^i - t^p\Big(\sum_{i=1}^{d} s_{d-i}(1-t)^{i-1}\Big)}{(1-t)^d}$$

where $d = n - c$ In the case where the ring $S/\mathcal{I}$ is Cohen-Macaulay, we have :

$$H_{S/\mathcal{I}}(t) = \frac{\sum_{i=0}^{p-1} \binom{c+i-1}{i} t^i}{(1-t)^d}$$

Definition 4 *For any non zero natural numbers c, p, we set*

$$h(c,p)(t) := \sum_{i=0}^{p-1} \binom{c+i-1}{i} t^i.$$

Remark that the $h-$vector of the polynomial $h(c,p)(t)$ is log concave, since for $i = 0, ..., p-3$, we have that

$$\binom{c+i-1}{i}\binom{c+i+1}{i+2} \leq (\binom{c+i}{i+1})^2.$$

Lemma 3 *For any non zero natural numbers c, p, we have the relation*

$$1 - h(c,p)(1-t)t^c = h(p,c)(t)(1-t)^p,$$

in particular $h(c,p)(t)\ (1-t)^c = 1 - h(p,c)(1-t)\ t^p$, $h(c,p)(t)\ (1-t)^c \equiv 1 \bmod t^p$.

Proof Let $\mathcal{I}$ be a square free monomial ideal having a $p-$linear resolution, such that $S/\mathcal{I}$ is a Cohen-Macaulay ring of codimension c, let $\mathcal{J} := \mathcal{I}^*$ be the Alexander dual of $\mathcal{I}$, it then follows that $S/\mathcal{J}$ is a Cohen-Macaulay ring of codimension p which has a $c-$linear resolution.

$$H_{S/\mathcal{I}}(t) = \frac{h(c,p)(t)}{(1-t)^{n-c}} = \frac{1 - B_{S/\mathcal{I}}(t)}{(1-t)^n}$$

$$H_{S/\mathcal{J}}(t) = \frac{h(p,c)(t)}{(1-t)^{n-p}} = \frac{1 - B_{S/\mathcal{J}}(t)}{(1-t)^n}$$

and by Alexander duality on the Hilbert series we have that : $1 - B_{S/\mathcal{I}}(t) = B_{S/\mathcal{J}}(1-t)$ but $h(c,p)(t)(1-t)^c = 1 - B_{S/\mathcal{I}}(t)$ and $h(p,c)(t)(1-t)^p = 1 - B_{S/\mathcal{J}}(t)$, so $B_{S/\mathcal{J}}(1-t) = 1 - h(p,c)(1-t)(t)^p$, so our claim follows from these identities.

Corollary 2 *Let $\mathcal{I} \subset S$ be any homogeneous ideal, $c = \operatorname{ht}(\mathcal{I}), d = n - c$ and p the smallest degree of a set of generators. Then we can write $H_{S/\mathcal{I}}(t)$ as follows*

$$H_{S/\mathcal{I}}(t) = \frac{h(c,p)(t) - t^p \Big(\sum_{i=1}^{\delta(\mathcal{I})} s_{\delta(\mathcal{I})-i} (1-t)^{i-1} \Big)}{(1-t)^d},$$

where the numbers $s_0, ..., s_{\delta(\mathcal{I})-1}$ are uniquely determined.

1. *Let $\mathcal{J}$ be a square free monomial ideal such that $S/\mathcal{J}$ is a Cohen-Macaulay ring of codimension p, let $\mathcal{I} := \mathcal{J}^*$ be the Alexander dual of $\mathcal{J}$, it then follows that $S/\mathcal{I}$ has a $p-$linear resolution. Let $c = \operatorname{codim}(S/\mathcal{I})$. Then*

$$H_{S/\mathcal{I}}(t) = \frac{h(c,p)(t) - t^p\Big(\sum_{i=1}^{d} s_{d-i}(1-t)^{i-1}\Big)}{(1-t)^{n-c}},$$

$$H_{S/\mathcal{J}}(t) = \frac{h(p,c)(t) + t^c\Big(\sum_{i=1}^{d} s_{d-i}t^{i-1}\Big)}{(1-t)^{n-p}}$$

2. *Let $\mathcal{I}$ be any square free monomial ideal $c = \operatorname{codim}(S/\mathcal{I})$, p the smallest degree of a set of generators of $\mathcal{I}$. Let $\mathcal{J} := \mathcal{I}^*$ be the Alexander dual of $\mathcal{I}$, then $p = \operatorname{codim}(S/\mathcal{J})$, c is the smallest degree of a set of generators of $\mathcal{J}$ and*

$$H_{S/\mathcal{I}}(t) = \frac{h(c,p)(t) - t^p\Big(\sum_{i=1}^{\delta(\mathcal{I})} s_{\delta(\mathcal{I})-i}(1-t)^{i-1}\Big)}{(1-t)^{n-c}},$$

$$H_{S/\mathcal{J}}(t) = \frac{h(p,c)(t) + t^c\Big(\sum_{i=1}^{\delta(\mathcal{I})} s_{\delta(\mathcal{I})-i}t^{i-1}\Big)}{(1-t)^{n-p}}.$$

Proof Since $1, t, ..., t^p, t^p(1-t), ..., t^p(1-t)^k, ...,$ are linearly independent the numbers s_i are uniquely defined.

$$H_{S/\mathcal{I}}(t) = \frac{h_{S/\mathcal{I}}(t)}{(1-t)^{n-c}} = \frac{1 - B_{S/\mathcal{I}}(t)}{(1-t)^n}$$
$$H_{S/\mathcal{J}}(t) = \frac{h_{S/\mathcal{J}}(t)}{(1-t)^{n-p}} = \frac{1 - B_{S/\mathcal{J}}(t)}{(1-t)^n}$$

by Alexander duality on the Hilbert series we have that :

$$B_{S/\mathcal{J}}(t) = 1 - B_{S/\mathcal{I}}(1-t) = (h(c,p)(1-t) - (1-t)^p\Big(\sum_{i=1}^{\delta(\mathcal{I})} s_{\delta(\mathcal{I})-i}t^{i-1}\Big))t^c,$$

but $h(c,p)(1-t)t^c = 1 - h(p,c)(t)\ (1-t)^p$, so

$$1 - B_{S/\mathcal{J}}(t) = (h(p,c)(t) + t^c\Big(\sum_{i=1}^{\delta(\mathcal{I})} s_{\delta(\mathcal{I})-i}t^{i-1}\Big))(1-t)^p.$$

This proves the claim.

Theorem 2 *1. For any $M-$vector $\mathbf{h}=(1,h_1,...)$ there exists Φ a $p-$Ferrer tableau such that h_i counts the number of elements in the $i-$diagonal of Φ.*

2. the $h-$vector of any $p-$regular ideal is the $h-$vector of a $p-$Ferrer ideal.

3. For any $M-$vector $\mathbf{h}=(1,h_1,...)$ we can explicitely construct a $p-$Ferrer tableau Φ such that $\mathbf{h}=(1,h_1,...)$ is the $h-$vector of $\mathcal{I}_\Phi^*$.

Proof

1. Let $\mathbf{h}=(1,h_1,...)$ be the $h-$vector of $S/\mathcal{J}$, by Macaulay, [S] 2.2 theorem h is obtained as the $M-$vector of a multicomplex Γ, where h_i counts the monomials of degree i in Γ. We establish a correspondence between multicomplex Γ and $p-$ Ferrer ideals:

   Suppose that Γ is a multicomplex in the variables $x_1,...,x_n$, to any monomial $x_1^{\alpha_1}...x_n^{\alpha_n}\in\Gamma$ we associated the vector $(\alpha_1+1,...,\alpha_n+1)\in(\mathbb{N}^*)^n$, let Φ be the image of Γ. By definition Γ is a multicomplex if and only if for any $u\in\Gamma$, and if v divides u then $v\in\Gamma$, this property is equivalent to the property:

   For any $(\alpha_1+1,...,\alpha_n+1)\in\Phi$ and $(\beta_1+1,...,\beta_n+1)\in(\mathbb{N}^*)^n$ such that $\beta_i\le\alpha_i$ for all i then $(\beta_1+1,...,\beta_n+1)\in\Phi$. That is Φ is a $p-$Ferrer tableau, such that h_i counts the number of elements in the $i-$diagonal of Φ.

2. Let $\mathcal{I}\subset S$ be any graded ideal with $p-$linear resolution, let $Gin(\mathcal{I})$ be the generic initial, by a theorem of Bayer and Stillman, $Gin(\mathcal{I})$ has a $p-$linear resolution, on the other hand they have the same Hilbert series, and from the remark in the introduction they have the same betti numbers. $Gin(\mathcal{I})$ is a monomial ideal, we can take the polarisation $P(Gin(\mathcal{I}))$, this is a square free monomial having $p-$linear resolution and the same betti numbers as $Gin(\mathcal{I})$, the Alexander dual $P(Gin(\mathcal{I}))^*$ is Cohen-Macaulay of codimension p , so there exists a Ferrer tableau Φ such that the $h-$vector of $S/P(Gin(\mathcal{I}))^*$ is the generating function of the diagonals of Φ, moreover the $h-$vector of $S/P(Gin(\mathcal{I}))^*$coincides with the $h-$vector of $S/(\mathcal{I}_\Phi)^*$. By the above proposition the $h-$vector of $S/P(Gin(\mathcal{I}))^*$ determines uniquely the $h-$vector of $S/P(Gin(\mathcal{I}))$, and the last one coincides with the $h-$vector of $S/\mathcal{I}_\Phi$.

3. Let recall from [S] how to associate to a $M-$vector $\mathbf{h}=(1,h_1,...,h_l)$ a multicomplex $\Gamma_{\mathbf{h}}$. For all $i\ge 0$ list all monomials in h_1 variables in reverse lexicographic order, let $\Gamma_{\mathbf{h},i}$ be set of first h_i monomials in this order, and $\Gamma_{\mathbf{h}}=\bigcup_{i=0}^{i=l}\Gamma_{\mathbf{h},i}$, in the first item we have associated to a multicomplex a $p-$Ferrer tableau Φ such that h_i is the number of elements in the $i-$diagonal of Φ. By the second item the $h-$vector of $S/(\mathcal{I}_\Phi)^*$ is exactly $\mathbf{h}$.

Example 4 *We consider the $h-$vector, $(1,4,3,4,1)$, following [S], this $h-$vector corresponds to the multicomplex*

$$1;x_1,...,x_4;x_1^2,x_1x_2,x_2^2;x_1^3,x_1^2x_2,x_1x_2^2,x_3^2;x_1^4,$$

and to the following $p-$Ferrer ideal $\mathcal{I}_\Phi$ generated by:

$$s_1t_1u_1v_1,$$

$$s_2t_1u_1v_1, s_1t_2u_1v_1, s_1t_1u_2v_1, s_1t_1u_1v_2,$$
$$s_3t_1u_1v_1, s_2t_2u_1v_1, s_1t_3u_1v_1,$$
$$s_4t_1u_1v_1, s_3t_2u_1v_1, s_2t_3u_1v_1, s_1t_4u_1v_1,$$
$$s_5t_1u_1v_1,$$

$\mathcal{I}_\Phi$ has the following prime decomposition:

$$(v_1, v_2) \cap (u_1, v_1) \cap (s_1, v_1) \cap (t_1, v_1) \cap (u_1, u_2) \cap (s_1, u_1) \cap (t_1, u_1) \cap$$
$$\cap (t_1, t_2, t_3, t_4) \cap (t_1, t_2 t_3, s_1) \cap (t_1, t_2, s_1, s_2) \cap (s_1, s_2, s_3, s_4, s_5)$$

and $\mathcal{I}_\Phi^*$ is generated by

$$v_1v_2, u_1v_1, s_1v_1, t_1v_1, u_1u_2, s_1u_1, t_1u_1,$$
$$t_1t_2t_3t_4, t_1t_2t_3s_1, t_1t_2s_1s_2, s_1s_2s_3s_4s_5$$

and the $h-$vector of $S/(\mathcal{I}_\Phi)^*$ is $(1, 4, 3, 4, 1)$.

4 Examples

Let $S = K[x_1, ..., x_n]$ be a polynomial ring. Let $\alpha \in \mathbb{N}^*$, for any element $P \in S$ we set $\widetilde{P}(x) = P(x_1^\alpha, ..., x_n^\alpha)$, more generally for any matrix with entries in S we set $\widetilde{M}$ be matrix obtained by changing the entry $P_{i,j}$ of M to $\widetilde{P}_{i,j}$.

Lemma 4 *Suppose that*

$$F^\bullet \ : \ 0 \to F_s \stackrel{M_s}{\to} F_{s-1} \to .... \to F_1 \stackrel{M_1}{\to} F_0 \to 0$$

is a minimal free resolution of a graded $S-$module M, then

$$\widetilde{F}^\bullet : 0 \to \widetilde{F}_s \stackrel{\widetilde{M}_s}{\to} \widetilde{F}_{s-1} \to .... \to \widetilde{F}_1 \stackrel{\widetilde{M}_1}{\to} \widetilde{F}_0 \to 0$$

is a minimal free resolution of a graded $S-$module $\widetilde{M}$. If $F^\bullet$ is a pure free resolution, that is $F_i = S^{\beta_i}(-a_i)$ for all $i = 0, ..., s$, then $\widetilde{F}^\bullet$ is also pure and $\widetilde{F}_i = S^{\beta_i}(-a_i\alpha)$ for all $i = 0, ..., s$.

Corollary 3 *1. Let $\Phi(p, c)$ be the $p-$ Ferrer diagram Cohen-Macaulay of codimension c. Let $\widetilde{\Phi}(p, c)$ be the Ferrer $p-$ Ferrer diagram obtained from $\Phi(p, c)$ by dividing any length unit into α parts, then the Alexander dual $\mathcal{I}^*_{\widetilde{\Phi}(p,c)}$ has a pure resolution of type $(0, c\alpha, ..., (c + p - 1)\alpha)$.*

2. *Let consider any sequence $0 < a_1 < a_2$, suppose that $\beta_0 - \beta_1 t^{a_1} + \beta_2 t^{a_2}$ is the Betti polynomial of a Cohen-Macaulay module of codimension 2, then we must have:*

$$a_1 = c\alpha, a_2 = (c+1)\alpha, \beta_1 = \frac{a_2}{a_2 - a_1}\beta_0, \beta_2 = \frac{a_1}{a_2 - a_1}\beta_0.$$

if $a_2 - a_1$ is a factor of a_2, a_1 then we can write

$$a_1 = c\alpha, a_2 = (c+1)\alpha, \beta_1 = (c+1)\beta_0, \beta_2 = c\beta_0,$$

with c a natural number. In particular a module obtaining by taking β_0 copies of $\mathcal{I}^*_{\widetilde{\Phi}(2,c)}$, has a pure resolution of type $(0, c\alpha, (c+1)\alpha)$.

Example 5 *Let $S = K[a,b,c]$, consider the free resolution of the algebra $S/(ab, ac, cd)$:*

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} a & 0 \\ -d & b \\ 0 & -c \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} cd & ac & ab \end{pmatrix}} S \longrightarrow 0$$

then we have a pure free resolution

$$0 \longrightarrow S^{2\beta_0} \xrightarrow{M_1} S^{3\beta_0} \xrightarrow{M_0} S^{\beta_0} \longrightarrow 0,$$

where

$$M_1 = \begin{pmatrix} a^\alpha & 0 & & & & \\ -d^\alpha & b^\alpha & & & & \\ 0 & -c^\alpha & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & a & 0 \\ & & & & -d^\alpha & b^\alpha \\ & & & & 0 & -c^\alpha \end{pmatrix}, M_0 = \begin{pmatrix} c^\alpha d^\alpha & a^\alpha c^\alpha & a^\alpha b^\alpha & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & cd & a^\alpha c^\alpha & a^\alpha b^\alpha \end{pmatrix},$$

with the obvious notation.

Example 6 *The algebra $S/(ab, ae, cd, ce, ef)$ has Betti-polynomial $1 - 5t^2 + 5t^3 - t^5$ but has not pure resolution.*

Example 7 Magic squares *Let S be a polynomial ring of dimension $n!$, It follows from [S] that the toric ring of $n \times n$ magic squares is a quotient $R_{\Phi_n} = S/\mathcal{I}_{\Phi_n}$, its $h-$polynomial is as follows:*

$$h_{R_{\Phi_n}}(t) = 1 + h_1 t + ... + h_l t^d,$$

where $h_1 = n! - (n-1)^2 - 1, d = (n-1)(n-2)$, If $n = 3$ we have $h_1 = 1, d = 2$, and there is no relation of degree two between two permutation matrices, but we have a degree three relation. Set

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

We can see that $M_1+M_2+M_3 = M_4+M_5+M_6$, so this relation gives a degree three generator in $\mathcal{I}_{\Phi_3}$, and in fact $\mathcal{I}_{\Phi_3}$ is generated by this relation. By using the cubic generators of $\mathcal{I}_{\Phi_3}$ we get cubic generators of $\mathcal{I}_{\Phi_n}$ for $n \geq 4$, but we have also quadratic generators, for example :

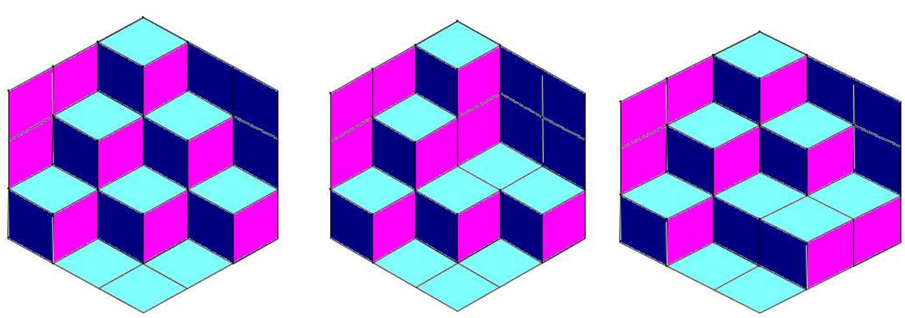
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So for $n \geq 4$, the smallest degree of a generator of the toric ideal $\mathcal{I}_{\Phi_n}$ is of degree 2. and unfortunately our proposition can give only information about h_1.

Example 8 *The Hilbert series of the following $p-$Ferrer tableaux are respectively:*

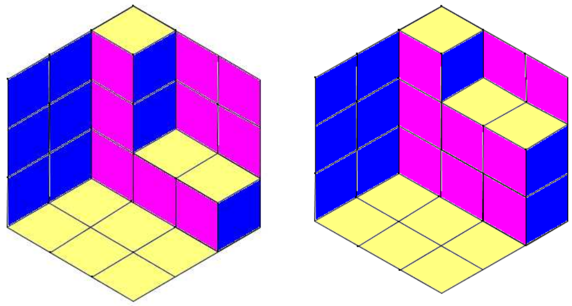
$$\frac{1+3t+6t^2}{(1-t)^6}, \quad \frac{1+2t+3t^2-5t^3}{(1-t)^7}, \quad \frac{1+3t+6t^2-t^3}{(1-t)^6}.$$

Let remark that $\dfrac{1+2t+3t^2-6t^3}{(1-t)^7} = \dfrac{1+3t+6t^2}{(1-t)^6}$.



Example 9 *The Hilbert series of the following $p-$Ferrer tableaux are respectively:*

$$H(t) = \frac{1+t+t^2-t^3(2+2(1-t))}{(1-t)^6}, H(t) = \frac{1+t+t^2-t^3(2+3(1-t)+(1-t)^2)}{(1-t)^6}$$



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